PG – 818

II Semester M.Sc. Degree Examination, June 2015
(RNS) (2011-12 and Onwards)
MATHEMATICS
M-203 : Functional Analysis

Time : 3 Hours Max. Marks : 80

Instructions : *1) Answer **any five** questions.*
*2) Choosing atleast **two** from **each** Part.*

PART – A

1. a) Define Banach space. Show that l_p^n is a Banach space under the norm $\| x \|^p = \left[\sum_{i=1}^{n} | x_i |^p \right]^{\frac{1}{p}}$ where p is the real number such that $1 < p < \infty$ and l_p^n is the space of all n-tuples $x = (x_1, x_2, ..., x_n)$ of scalars. **6**

   b) Let M be a closed linear subspace of a normed linear space N. Show that the quotient space $\frac{N}{M}$ is also normed linear space. Further if N is a Banach space then prove that $\frac{N}{M}$ is a Banach space. **10**

2. a) Show that every closed subspace of a Banach space is Banach space. **6**

   b) If N and N' are normed linear spaces and $T : N \to N'$, then show that the following are equivalent

   i) $\| T \| = \sup \left\{ \frac{\| T(x) \|}{\| x \|} : x \in N, x \neq 0 \right\}$

   ii) $\| T \| = \sup \{ \| T(x) \| : x \in N, \| x \| \leq 1 \}$

   iii) $\| T \| = \sup \{ \| T(x) \| : x \in N, \| x \| = 1 \}$. **10**

P.T.O.

3. a) Let M be a linear subspace of a normed linear space N and let f be a continuous linear functional defined on M. If x_0 is a vector not in M and if $M_0 = M + [x_0]$ denotes the linear subspace spanned by M and x_0, then prove that f can be extended to a continuous linear functional f_0 defined on M_0 such that $\| f_0 \| = \| f \|$. **10**

   b) If N is a normed linear space and x_0 is a nonzero vector in N, then prove that there exists a functional f_0 in N* such that $f_0(x_0) = \| x_0 \|$ and $\| f_0 \| = 1$. **6**

4. a) Let B be a Banach space and N be a normed linear space. If $\{T_i\}_{i \in I}$ is a nonempty set of continuous linear transformations of B into N with the property that $\{T_i(x)\}_{i \in I}$ is a bounded subset of N for each x in B, then prove that $\{T_i\}_{i \in I}$ is a bounded subset of B(B, N). **7**

   b) Prove that a nonempty subset X of a normed linear space N is bounded $\Leftrightarrow$ f(X) is a bounded set of numbers for each f in N*. **5**

   c) Let B be a Banach space and N be a normed linear space. If $\{T_n\}$ is a sequence in B(B, N) such that $T(x) = \lim T_n(x)$ exists for each x in B, then prove that T is a continuous linear transformation. **4**

PART – B

5. a) If M is a proper closed linear subspace of a Hilbert space H then prove that there exists a non-zero vector z_0 in H such that $z_0 \perp M$. **6**

   b) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$ then prove that the linear subspace M + N is also closed. **6**

   c) If M is a closed linear subspace of a Hilbert space H then prove that $H = M \oplus M^{\perp}$. **4**

6. a) Let H be a Hilbert space and let f be an arbitrary functional in H*. Then prove that there exists a unique vector y in H such that $f(x) = \langle x, y \rangle$ for every x in H. **5**

   b) Show that there exists an antilinear norm preserving isometric isomorphism between a Hilbert space H and its dual H*. **5**

   c) Prove that every non zero Hilbert space contains a complete orthonormal set. **6**

7. a) Prove that the set of all normal operators on a Hilbert space H forms a closed subset of B(H). **6**

b) Prove that an operator T on a Hilbert space H is normal if and only if the pure and imaginary Parts of T commute each other. **6**

c) Show that a unitary operator on a Hilbert space preserves both the inner product and the norm. **4**

8. a) Prove each of the following for an operator T on a Hilbert space H.

i) T is self adjoint $\Leftrightarrow$ $\langle Tx, x \rangle$ is real for each vector x.

ii) T is normal $\Leftrightarrow$ $\| T^* x \| = \| Tx \|$ for each x.

iii) T is unitary $\Leftrightarrow$ T is an isometric isomorphism of H into itself. **6**

b) Let $P_1, P_2, \ldots, P_n$ be projections on the closed linear subspaces $M_1, M_2, \ldots, M_n$ of a Hilbert space H and $M = M_1 + M_2 + \ldots + M_n$. Then show that $P = P_1 + P_2 + \ldots + P_n$ is a projection if and only if P_i's are pairwise orthogonal and in this case P is the projection on M. **7**

c) If P and Q are the projections on closed linear subspaces M and N of a Hilbert space H, prove that PQ is a projection if and only if PQ = QP. In this case show that PQ is the projection on $M \cap N$. **3**